

DEPARTMENT OF MATHEMATICS
MAT391 ENGINEERING MATHEMATICS III
WEEK 1: LECTURE NOTE

LAPLACE TRANSFORMS

INTRODUCTION

Laplace transform methods have a key role to play in the modern approach to the analysis and design of engineering systems. The stimulus for developing these methods was the pioneering work of the English electrical engineer **Oliver Heaviside** (1850-1925) in developing a method for the systematic solution of ordinary differential equations with constant coefficients. Heaviside was concerned with solving practical problems.

The standard methods of solving second-order differential equations with constant coefficients,

$$ay'' + by' + cy = f(x), \quad (1)$$

are either by substitution of an assumed solution or by using D-operator methods. In either case, the general solution is first ascertained and the arbitrary constants evaluated by insertion of the initial conditions.

Laplace transform:

1. An algebraic processes by which the solution of the differential equation is obtained.
2. The initial conditions are involved from the early stages so that the determination of the particular solution is considerably shortened.
3. Enables us to deal with situations where the function is discontinuous: classical methods necessarily require the function to be continuous.
4. The use of Laplace transforms is a powerful tool and has applications in numerous fields of technology.

Definition and notation of Laplace transform

Definition 1. The **Laplace transform** of an expression $f(t)$ is denoted by $L[f(t)]$ (or $L\{f(t)\}$) and is defined as the semi-infinite integral

$$L[f(t)] = \int_0^{\infty} f(t)e^{-st} dt = F(s) \quad (2)$$

The parameter s is assumed to be positive and large enough to ensure that the integral converges; that is,

$$\lim_{t \rightarrow \infty} f(t)e^{-st} = 0$$

for most common functions $f(t)$.

The symbol L denotes the Laplace transform operator; when it operates on a function $f(t)$, it transforms it into a function $F(s)$ of the variable s . We say the operator transforms the function $f(t)$ in the t domain (usually called the time domain) into the function $F(s)$ in the s domain (usually called the frequency domain).

Laplace transforms of simple functions

Example 2 (Laplace transform of a constant function). Find the Laplace transform of $f(t) = a$ for a constant a .

$$\begin{aligned} F(s) &= L[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty ae^{-st}dt \\ &= a \int_0^\infty e^{-st}dt = \frac{a}{s}, \quad \text{for } s > 0. \end{aligned}$$

Example 3 (Laplace transform of natural Exponential function). Find the Laplace transform of $f(t) = e^{at}$ for a constant a .

$$\begin{aligned} F(s) &= L[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{at}e^{-st}dt \\ &= \int_0^\infty e^{-(s-a)t}dt = \frac{1}{s-a}, \quad \text{for } s > a. \end{aligned}$$

Example 4 (Laplace transform of $\cos at$ and $\sin at$). By Euler's formula we know that

$$e^{iat} = \cos at + i \sin at$$

so that $\cos at = \operatorname{Re}(e^{iat})$ and $\sin at = \operatorname{Im}(e^{iat})$. It follows from Example 3 that

$$L[e^{iat}] = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i \frac{a}{s^2+a^2}.$$

Therefore,

$$L[\cos at] = \operatorname{Re}(L[e^{iat}]) = \frac{s}{s^2+a^2} \quad (3)$$

$$L[\sin at] = \operatorname{Im}(L[e^{iat}]) = \frac{a}{s^2+a^2} \quad (4)$$

Example 5 (Laplace transform of $\cosh(at)$ and $\sinh(at)$). By the definition of hyperbolic functions we have that

$$\begin{aligned} \cosh at &= \frac{1}{2}[e^{at} + e^{-at}] \\ \sinh at &= \frac{1}{2}[e^{at} - e^{-at}]. \end{aligned}$$

It follows from Example 3 that

$$L[\cosh at] = \frac{1}{2} \int_0^\infty [e^{-(s-a)t} + e^{-(s+a)t}]dt = \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right] = \frac{s}{s^2-a^2}$$

and

$$L[\sinh at] = \frac{1}{2} \int_0^\infty [e^{-(s-a)t} - e^{-(s+a)t}]dt = \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right] = \frac{a}{s^2-a^2}$$

Example 6 (Laplace transform of $f(t) = t^n$).

$$\begin{aligned} L[t] &= \int_0^\infty te^{-st}dt = \left[-\frac{t}{s}e^{-st} \right]_0^\infty + \frac{1}{s} \int_0^\infty e^{-st}dt = \frac{1}{s^2} \\ L[t^2] &= \int_0^\infty t^2e^{-st}dt = \left[-\frac{t^2}{s}e^{-st} \right]_0^\infty + \frac{2}{s} \int_0^\infty te^{-st}dt = \frac{2!}{s^3} \\ L[t^3] &= \int_0^\infty t^3e^{-st}dt = \left[-\frac{t^3}{s}e^{-st} \right]_0^\infty + \frac{3}{s} \int_0^\infty t^2e^{-st}dt = \frac{3}{s} \left[\frac{2!}{s^3} \right] = \frac{3!}{s^4} \\ L[t^4] &= \int_0^\infty t^4e^{-st}dt = \left[-\frac{t^4}{s}e^{-st} \right]_0^\infty + \frac{4}{s} \int_0^\infty t^3e^{-st}dt = \frac{4}{s} \left[\frac{3!}{s^4} \right] = \frac{4!}{s^5} \\ &\vdots \\ L[t^n] &= \int_0^\infty t^ne^{-st}dt = \left[-\frac{t^n}{s}e^{-st} \right]_0^\infty + \frac{n}{s} \int_0^\infty t^{n-1}e^{-st}dt = \frac{n!}{s^{n+1}} \end{aligned}$$

The following Laplace transforms are consequences of Examples 2-6:

$$L[4] = \frac{4}{s}, \quad s > 0. \quad (5)$$

$$L[t^7] = \frac{7!}{s^8}, \quad s > 0. \quad (6)$$

$$L[e^{3t}] = \frac{1}{s-3}, \quad s > 3. \quad (7)$$

$$L[e^{-2t}] = \frac{1}{s+2}, \quad s > -2. \quad (8)$$

$$L[\cos 2t] = \frac{s}{s^2 + 4}, \quad s \in \mathbb{R}. \quad (9)$$

$$L[\sin \pi t] = \frac{\pi}{s^2 + \pi^2}, \quad s \in \mathbb{R}. \quad (10)$$

$$L[\sinh 3t] = \frac{3}{s^2 - 9}, \quad s > 3. \quad (11)$$

$$L[\cosh(-2t)] = \frac{s}{s^2 - 4}, \quad s > 2. \quad (12)$$

Properties of the Laplace transform

Theorem 7 (THE LINEARITY PROPERTY). *If $f(t)$ and $g(t)$ are functions having Laplace transforms and if α and β are any real constants then*

$$L[\alpha f(t) + \beta g(t)] = \alpha L[f(t)] + \beta L[g(t)].$$

Theorem 8 (THE FIRST SHIFT PROPERTY). *If $f(t)$ is a function having Laplace transform $F(s)$, with $s > \sigma$, then the function $e^{at}f(t)$ also has a Laplace transform, given by*

$$L[e^{at}f(t)] = F(s-a), \quad s > \sigma + a.$$

Theorem 9 (DERIVATIVE-OF-TRANSFORM PROPERTY). *If $f(t)$ is a function having Laplace transform $F(s) = L[f(t)]$, $s > \sigma$, then the functions $t^n f(t)$ ($n = 1, 2, \dots$) also have Laplace transforms, given by*

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [F(s)] \quad s > \sigma.$$

Theorem 10 (Dividing by t). *If $f(t)$ is a function having Laplace transform $F(s) = L[f(t)]$, then*

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(\sigma) d\sigma,$$

provided that $\lim_{t \rightarrow 0} \frac{f(t)}{t}$ exists.

Example 11. *Use of the properties of Laplace transform*

1. Find Laplace transform of $f(t) = 3t + 2e^{3t}$.

$$\begin{aligned} F(s) &= L[3t + 2e^{3t}] \\ &= 3L[t] + 2L[e^{3t}] \\ &= \frac{3}{s^{1+1}} + 2\left(\frac{1}{s-3}\right) \\ &= \frac{3}{s^2} + \frac{2}{s-3} \end{aligned}$$

2. Find Laplace transform of $f(t) = 3 + 5e^{-2t}$.

$$\begin{aligned} F(s) &= L[3 + 5e^{-2t}] \\ &= L[3] + 5L[e^{-2t}] \\ &= \frac{3}{s} + 5 \left(\frac{1}{s - (-2)} \right) \\ &= \frac{3}{s} + \frac{5}{s + 2} \end{aligned}$$

3. Find Laplace transform of $f(t) = 3 \sin 4t - 2t^3$.

$$\begin{aligned} F(s) &= L[3 \sin 4t - 2t^3] \\ &= 3L[\sin 4t] - 2L[t^3] \\ &= 3 \left[\frac{4}{s^2 + 16} \right] - 2 \left[\frac{3!}{s^4} \right] \\ &= 12 \left[\frac{1}{s^2 + 16} - \frac{1}{s^4} \right] \end{aligned}$$

4. Find Laplace transform of $f(t) = \sin 2t \cos 2t + e^t - 7$.

$$\begin{aligned} F(s) &= L[\sin 2t \cos 2t + e^t - 7] \\ &= L[\sin 2t \cos 2t] + L[e^t] - L[7] \\ &= L\left[\frac{1}{2} \sin 4t\right] + \frac{1}{s-1} - \frac{7}{s} = \frac{1}{2} \{L[\sin 4t]\} + \frac{1}{s-1} - \frac{7}{s} \\ &= \frac{1}{2} \left\{ \frac{4}{s^2 + 16} \right\} + \frac{1}{s-1} - \frac{7}{s} = \frac{2}{s^2 + 16} + \frac{1}{s-1} - \frac{7}{s} \end{aligned}$$

5. Find Laplace transform of $f(t) = t^2 \sinh 2t$.

Solution. Because

$$F(s) = L[\sinh 2t] = \frac{2}{s^2 - 4}$$

we have

$$L[t^2 \sinh 2t] = (-1)^2 \frac{d^2}{ds^2} [F(s)] = (-1)^2 \frac{d^2}{ds^2} \left\{ \frac{2}{s^2 - 4} \right\} = \frac{4(4 + 3s^2)}{(s^2 - 4)^3}$$

6. Find Laplace transform of $f(t) = e^{3t} \sin t$.

Solution. Because $F(s) = L[\sin t] = \frac{1}{s^2 + 1}$ we have

$$L[e^{3t} \sin t] = F(s - 3) = \frac{1}{(s - 3)^2 + 1}$$

7. Find Laplace transform of $f(t) = \frac{e^{2t} - 1}{t}$.

Solution. Because $F(s) = L[e^{2t} - 1] = \frac{1}{s - 2} - \frac{1}{s}$ and $\lim_{t \rightarrow 0} \frac{e^{2t} - 1}{t} = 1$, we have

$$\begin{aligned} L\left[\frac{e^{2t} - 1}{t}\right] &= \int_s^\infty F(\sigma) d\sigma \\ &= \int_s^\infty \left[\frac{1}{\sigma - 2} - \frac{1}{\sigma} \right] d\sigma \\ &= \lim_{\sigma \rightarrow \infty} \left[\ln \left(\frac{\sigma - 2}{\sigma} \right) - \ln \left(1 - \frac{2}{\sigma} \right) \right] = \ln 1 - \ln \left(1 - \frac{2}{s} \right) \\ &= \ln \left(\frac{s}{s - 2} \right) \end{aligned}$$

Table of Laplace transforms

It is appropriate at this stage to draw together the results proved above for simple functions for easy access.

$f(t)$	$L[f(t)] = F(s)$	Region of convergence
a	$\frac{a}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
$t^n \quad n = 2, 3, 4, \dots$	$\frac{n!}{s^{n+1}}$	$s > 0$
e^{at}	$\frac{1}{s - a}$	$s > a$
$\cos at$	$\frac{s}{s^2 + a^2}$	$s > a$
$\sin at$	$\frac{a}{s^2 + a^2}$	$s > a$
$\cosh at$	$\frac{s}{s^2 - a^2}$	$s > a $
$\sinh at$	$\frac{a}{s^2 - a^2}$	$s > a $
$e^{-at} \cos bt$	$\frac{s + a}{(s + a)^2 + b^2}$	$s > -a$
$e^{-at} \sin bt$	$\frac{b}{(s + a)^2 + b^2}$	$s > -a$